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LOGISTIC ORDER STATISTICS AND THEIR PROPERTIES†

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LOGISTIC ORDER STATISTICS AND THEIR PROPERTIES[†]

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1. INTRODUCTION

Let $Z_1, Z_2, \ldots, Z_n$ be a random sample of size n from the logistic $L(0, \frac{\pi^2}{3})$ population with probability density function

$$f^*(z) = e^{-z}/(1+e^{-z})^2, \qquad -\infty < z < \infty, \tag{1.1}$$

and cumulative distribution function

$$F^*(z) = 1/(1+e^{-z}), \qquad -\infty < z < \infty. \tag{1.2}$$

Let $Z_{1:n} \leq Z_{2:n} \leq \ldots \leq Z_{n:n}$ denote the order statistics obtained by arranging the above sample in increasing order of magnitude. Then the density function of $Z_{i:n}$ $(1 \leq i \leq n)$ is given by

$$f^*_{i:n}(z_i) = \frac{n!}{(i-1)!(n-i)!}\left\{F^*(z_i)\right\}^{i-1}\left\{1-F^*(z_i)\right\}^{n-i} f^*(z_i),$$

$$-\infty < z_i < \infty, \tag{1.3}$$

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and the joint density function of $Z_{i:n}$ and $Z_{j:n}$ $(1 \le i < j \le n)$ is given by

$$f^*_{i,j:n}(z_i,z_j) = \frac{n!}{(i-1)!(j-i-1)!(n-j)!}\left\{F^*(z_i)\right\}^{i-1}\left\{F^*(z_j)-F^*(z_i)\right\}^{j-i-1}\left\{1-F^*(z_j)\right\}^{n-j}$$

$$f^*(z_i)\, f^*(z_j), \qquad -\infty<z_i<z_j<\infty. \tag{1.4}$$

Let us now denote the single moments $E(Z^k_{i:n})$ by $\alpha^{*(k)}_{i:n}$ for $1 \le i \le n$, $k \ge 1$, and the product moments $E(Z_{i:n}Z_{j:n})$ by $\alpha^*_{i,j:n}$ for $1 \le i < j \le n$. For convenience, let us also denote $E(Z_{i:n})$ by $\alpha^*_{i:n}$ and $E(Z^2_{i:n})$ by $\alpha^*_{i,i:n}$ for $1 \le i \le n$. Further, let us denote $\mathrm{Cov}(Z_{i:n},Z_{j:n})$ by $\beta^*_{i,j:n}$.

Order statistics $Z_{i:n}$ and their moments have been studied in great detail by several authors including Birnbaum and Dudman (1963), Gupta and Shah (1965), Tarter and Clark (1965), Shah (1966, 1970), Gupta et al. (1967), Malik (1980), George and Rousseau (1987), and Balakrishnan and Malik (1990). Birnbaum and Dudman (1963) derived explicit expression for the cumulants of order statistics and tabulated the means and standard deviations for sample sizes up to ten and for some large sample sizes as well. They then summarized these quantities in graphs to facilitate interpolation to other sample sizes. Gupta and Shah (1965) derived exact expressions for the moments of order statistics in terms of Bernoulli and Stirling numbers of first kind and used them to tabulate the first four moments for sample sizes up to ten. They also expressed the cumulants in terms of polygamma functions, as was originally pointed out by Plackett (1958). It should be mentioned here that Plackett (1958) used these explicit expressions of the moments of logistic order statistics to develop a method of approximating the moments of order statistics from an arbitrary continuous distribution. Distribution of the sample range has been studied by Gupta and Shah (1965) who also provided a short table of its percentage points for n=2 and 3. By generalizing this result, Malik (1980) derived the exact formula for the cumulative distribution function of the r^{th} quasi–range, viz.,

$Z_{n-r:n}-Z_{r+1:n}$ for $r=0, 1, \ldots, \left[\frac{n-1}{2}\right]$. In an independent study, Tarter and Clark (1965) reproduced some of the results of Gupta and Shah (1965) and then studied the distribution of the sample median in detail. George and Rousseau (1987) recently examined the distribution of the sample midrange, viz., $(Z_{1:n}+Z_{n:n})/2$, and established several relationships in distribution between the midrange and sample median of the logistic and Laplace random variables. A series expression for the covariance of two order statistics has been provided by both Gupta and Shah (1965) and Tarter and Clark (1965). Shah (1966) tabulated the covariances for sample sizes up to ten and Gupta, Qureishi and Shah (1967) extended this table for sample sizes up to twenty five. It should be mentioned here, however, that by means of some recurrence formulas Kjelsberg (1962) had already derived exact numerical results for the covariances from samples of size five or less.

By using the fact that $f^*(z)$ and $F^*(z)$ given in Eqs. (1.1) and (1.2) satisfy the relation

$$f^*(z) = F^*(z)\left\{1-F^*(z)\right\}, \tag{1.5}$$

Shah (1966, 1970) established several recurrence relations satisfied by the single and the product moments of order statistics. Recently, Balakrishnan and Malik (1990) prepared tables of means, variances and covariances for sample sizes up to [illegible] by applying these relations in a simple and systematic recursive way. Some of the results in the references cited above have also been summarized in a review article by Malik (1985).

The truncated logistic distribution plays a role in a variety of applications, as has been mentioned by Kjelsberg (1962). Order statistics and their moments from a general truncated logistic distribution have been studied by Tarter (1966). He derived exact and explicit expressions for the means, variances and covariances of order statistics in terms

of a finite series involving logarithms and dilogarithms of the constants of truncation. By following the lines of Shah (1966, 1970), Balakrishnan and Joshi (1983 a,b) established several recurrence relations satisfied by the single and the product moments of order statistics from a symmetrically truncated logistic distribution. Then, Balakrishnan and Kocherlakota (1986) generalized these results to the doubly truncated logistic distribution and displayed that these recurrence relations could be used systematically in order to evaluate the means, variances and covariances of all order statistics for all sample sizes.

In this paper we present a detailed discussion of order statistics from the logistic distribution and some of their properties. In Section 2 we give the percentage points and modes of order statistics. In Section 3 we derive exact and explicit expressions for the single and the product moments of order statistics. These work in terms of gamma function and its successive derivatives. In Section 4 we present some recurrence relations satisfied by the single and the product moments of order statistics which would enable one to compute the means, variances and covariances in a simple recursive way. The distribution function of the sample range, as derived by Gupta and Shah (1965), is presented in Section 5 and the distribution of the r^{th} quasi–range derived by Malik (1980) is also given here for the sake of completeness. In Section 6 we give some relations between moments for the case of the doubly truncated logistic distribution that are due to Balakrishnan and Kocherlakota (1986). In Section 7 we present details of tables that are available in this context. Finally, in Section 8 we describe Plackett's (1958) method of approximating the moments of order statistics from an arbitrary continuous distribution by using the moments of logistic order statistics.

2. PERCENTAGE POINTS AND MODES

The distribution function of $Z_{1:n}$ is given by

$$F^*_{1:n}(z_1) = 1-\left\{1-F^*(z_1)\right\}^n . \tag{2.1}$$

From (2.1) we obtain the 100α percentage point of $Z_{1:n}$ to be

$$z_{1:n(\alpha)} = \ln\left\{1-(1-\alpha)^{1/n}\right\} - \ln(1-\alpha)^{1/n} \quad , 0<\alpha<1. \tag{2.2}$$

Next, the distribution function of $Z_{n:n}$ is given by

$$F^*_{n:n}(z_n) = \left\{F^*(z_n)\right\}^n. \tag{2.3}$$

From (2.3) we obtain the 100α percentage point of $Z_{n:n}$ to be

$$z_{n:n(\alpha)} = \ln(\alpha^{1/n}) - \ln(1-\alpha^{1/n}), \quad 0<\alpha<1. \tag{2.4}$$

Similarly, the distribution function of $Z_{i:n}$ $(2\le i\le n-1)$ is given by

$$\begin{aligned} F^*_{i:n}(z_i) &= I_{F^*(z_i)}(i,n-i+1) \\ &= \frac{1}{B(i,n-i+1)} \int_0^{F^*(z_i)} u^{i-1}\,(1-u)^{n-i}\,du. \end{aligned} \tag{2.5}$$

Now, let $B_\alpha(i,n-i+1)$ denote the 100α percentage point of the Beta$(i,n-i+1)$ distribution. From (2.5) we then have the 100α percentage point of $Z_{i:n}$ $(2\le i\le n-1)$ to be

$$z_{i:n(\alpha)} = \ln B_\alpha(i,n-i+1) - \ln\{1-B_\alpha(i,n-i+1)\}, \quad 0<\alpha<1. \tag{2.6}$$

By using the symmetric relation satisfied by the incomplete beta functions, we observe from Eqs. (2.2), (2.4) and (2.6) that the 100α percentage point of $Z_{i:n}$ is simply the negative of $100(1-\alpha)$ percentage point of $Z_{n-i+1:n}$. While the percentage points of $Z_{1:n}$ and $Z_{n:n}$ may be obtained easily from (2.2) and (2.4), respectively, the percentage points of $Z_{i:n}$ for $2 \le i \le n-1$ may be obtained from (2.6) either by using the extensive tables of incomplete beta function prepared by Karl Pearson (1934) and Pearson and Hartley (1970) or by using the algorithm given by Cran, Martin and Thomas (1977). Gupta and Shah (1965) have tabulated some percentage points of all order statistics for sample sizes up to 10 and some selected order statistics for sample sizes up to 25.

Next, by differentiating the density function of $Z_{i:n}$ in (1.3) with respect to z_i and using the relation in (1.5), we get

$$\frac{df^*_{i:n}(z_i)}{dz_i} = \frac{n!}{(i-1)!(n-i)!} \left\{F^*(z_i)\right\}^{i-1} \left\{1-F^*(z_i)\right\}^{n-i} \left[i-(n+1)F^*(z_i)\right] . \tag{2.7}$$

Upon equating (2.7) to zero and solving for z_i, we obtain the mode of $Z_{i:n}$ $(1 \le i \le n)$ to be

$$m^*_{i:n} = \ln\left\{i/(n-i+1)\right\} = \ln\,(p_i/q_i) , \tag{2.8}$$

where $p_i=1-q_i=i/(n+1)$. Due to the symmetry of the logistic distribution, we observe once again that the mode of $Z_{i:n}$ is simply negative of the mode of $Z_{n-i+1:n}$.

3. MOMENTS AND CUMULANTS

From (1.3) we have the moment generating function of $Z_{i:n}$ $(1 \le i \le n)$ to be

$$M^*_{i:n}(t) = E\left\{e^{tZ_{i:n}}\right\}$$

$$= \frac{1}{B(i,n-i+1)} \int_{-\infty}^{\infty} \frac{e^{-(n-i+1)z+tz}}{(1+e^{-z})^{n+1}} dz$$

$$= B(i+t,n-i+1-t)/B(i,n-i+1)$$

$$= \frac{\Gamma(i+t)}{\Gamma(i)} \frac{\Gamma(n-i+1-t)}{\Gamma(n-i+1)}, \tag{3.1}$$

where $B(.,.)$ and $\Gamma(.)$ are the usual complete beta and gamma functions, respectively. An alternate expression of $M^*_{i:n}(t)$ involving Bernoulli numbers and Stirling numbers of first kind has been given by Gupta and Shah (1965). From the expression of the moment generating function in (3.1), we obtain the following:

$$\alpha^*_{i:n} = E(Z_{i:n}) = \psi(i) - \psi(n-i+1), \tag{3.2}$$

$$\alpha^{*(2)}_{i:n} = E(Z^2_{i:n}) = \psi'(i) + \psi'(n-i+1) + \left\{\psi(i) - \psi(n-i+1)\right\}^2, \tag{3.3}$$

and

$$\beta^*_{i,i:n} = \mathrm{Var}(Z_{i:n}) = \psi'(i) + \psi'(n-i+1), \tag{3.4}$$

where

$$\psi(z) = \frac{d}{dz} \ln\Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}$$

and

$$\psi'(z) = \frac{d^2}{dz^2} \ln\Gamma(z) = \frac{\Gamma''(z)}{\Gamma(z)} - \psi^2(z)$$

are the digamma and trigamma functions, respectively. Thus, from Eqs. (3.2) – (3.4), one may compute the means and variances of order statistics either by using the extensive tables of digamma and trigamma functions prepared by Davis (1935) and Abramowitz and Stegun (1965) or by using the algorithms given by Bernardo (1976) and Schneider (1978). Gupta and Shah (1965) have given exact expressions for the first four moments of order statistics for sample sizes up to 10 and the values of mean and variance have been tabulated recently by Balakrishnan and Malik (1990) for sample sizes up to 50. We may note here that the moment generating function in (3.1) may be used to obtain higher order single moments also by involving polygamma functions.

From (3.1) we get the cumulant generating function of $Z_{i:n}$ $(1 \le i \le n)$ to be

$$\begin{aligned} K^*_{i:n}(t) &= \ln M^*_{i:n}(t) \\ &= \ln \Gamma(i+t) + \ln \Gamma(n-i+1-t) - \ln \Gamma(i) - \ln \Gamma(n-i+1). \qquad (3.5) \end{aligned}$$

From (3.5) we obtain the k^{th} cumulant of $Z_{i:n}$ $(1 \le i \le n)$ to be

$$\begin{aligned} \kappa^{*(k)}_{i:n} &= \frac{d^k}{dt^k} \ln \Gamma(i+t)\Big|_{t=0} + \frac{d^k}{dt^k} \ln \Gamma(n-i+1-t)\Big|_{t=0} \\ &= \psi^{(k-1)}(i) + (-1)^k\, \psi^{(k-1)}(n-i+1), \qquad (3.6) \end{aligned}$$

where

$$\psi^{(k-1)}(z) = \frac{d^k}{dz^k} \ln\Gamma(z) \qquad \text{for } k=1,2,\ldots,$$

and

$$\psi^{(0)}(z) \equiv \psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} .$$

It is clear from (3.6) that for k=1,2,...,

$$\kappa_{i:n}^{*(2k-1)} = -\kappa_{n-i+1:n}^{*(2k-1)} \tag{3.7}$$

and

$$\kappa_{i:n}^{*(2k)} = \kappa_{n-i+1:n}^{*(2k)}. \tag{3.8}$$

These may also be observed simply by using the symmetry of the logistic distribution. By applying the series expansions for $\psi(z)$ and $\psi^{(k-1)}(z)$ given by

$$\psi(z) = \sum_{\ell=1}^{\infty} \left\{\frac{1}{\ell} - \frac{1}{\ell+z-1}\right\}$$

and

$$\psi^{(k-1)}(z) = (-1)^k (k-1)! \sum_{\ell=1}^{\infty} \frac{1}{(\ell+z-1)^k}, \; k \geq 2,$$

we obtain from (3.6) that for n–i+1>i

$$\kappa_{i:n}^{*} \equiv \kappa_{i:n}^{*(1)} = -\left\{\frac{1}{i} + \frac{1}{i+1} + \dots + \frac{1}{n-i}\right\} \tag{3.9}$$

and

$$\kappa_{i:n}^{*(k)} = (-1)^k (k-1)! \left\{\sum_{\ell=1}^{\infty} \frac{1}{(\ell+i-1)^k} + (-1)^k \sum_{\ell=1}^{\infty} \frac{1}{(\ell+n-i)^k}\right\}. \tag{3.10}$$

The above formulae for the first four cumulants were originally given by Plackett (1958). From (3.9) we get

$$\kappa_{1:n}^{*} = -\left\{1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n-1}\right\}$$

which was also given by Gumbel (1958). The exact and explicit expression for the cumulants of logistic order statistics given in Eqs. (3.9) and (3.10) will later be used in Section 8 for developing some series approximations for the moments of order statistics from an arbitrary continuous distribution.

From the joint density of $Z_{i:n}$ and $Z_{j:n}$ $(1 \le i < j \le n)$ in (1.4), we have the joint moment generating function of $Z_{i:n}$ and $Z_{j:n}$ to be

$$\begin{aligned} M^*_{i,j:n}(t_1,t_2) &= E\left\{e^{t_1 Z_{i:n} + t_2 Z_{j:n}}\right\} \\ &= \frac{n!}{(i-1)!(j-i-1)!(n-j)!} \int_{-\infty}^{\infty} \int_{-\infty}^{z_j} e^{t_1 z_i + t_2 z_j} \left\{F^*(z_i)\right\}^{i-1} \\ &\qquad \left\{F^*(z_j) - F^*(z_i)\right\}^{j-i-1} \left\{1-F^*(z_j)\right\}^{n-j} \\ &\qquad f^*(z_i)\, f^*(z_j)\, dz_i\, dz_j. \end{aligned} \tag{3.11}$$

Making the transformations

$$u = F^*(z_i) = \frac{1}{1+e^{-z_i}} \quad \text{and} \quad v = F^*(z_j) = \frac{1}{1+e^{-z_j}}$$

and thence, noting that

$$e^{z_i} = \frac{u}{1-u} \quad \text{and} \quad e^{z_j} = \frac{v}{1-v},$$

we can rewrite Eq. (3.11) as

$$M^*_{i,j:n}(t_1,t_2) = \frac{n!}{(i-1)!(j-i-1)!(n-j)!}\int_0^1\int_0^v \frac{u^{t_1}}{(1-u)^{t_1}}\ \frac{v^{t_2}}{(1-v)^{t_2}}$$

$$u^{i-1}\,(v-u)^{j-i-1}\,(1-v)^{n-j}\,du\,dv.$$

By expanding $(1-u)^{-t_1}$ as an infinite series in powers of u, we obtain

$$M^*_{i,j:n}(t_1,t_2) = \frac{n!}{(i-1)!(j-i-1)!(n-j)!}\sum_{\ell=0}^{\infty}\frac{(t_1+\ell-1)^{(\ell)}}{\ell!}\int_0^1\int_0^v$$

$$u^{t_1+i-1+\ell}(v-u)^{j-i-1}\,v^{t_2}(1-v)^{n-j-t_2}\,du\,dv, \qquad (3.12)$$

where

$$(t_1+\ell-1)^{(\ell)} = 1 \qquad \text{if } \ell=0$$

$$= t_1(t_1+1)\ldots..\,(t_1+\ell-1) \qquad \text{if } \ell\geq 1.$$

By noting that

$$\int_0^v u^{t_1+i-1+\ell}(v-u)^{j-i-1}\,du = v^{j+t_1+\ell-1}\;B(t_1+i+\ell,j-i),$$

we may rewrite Eq. (3.12) as

$$M^{*}_{i,j:n}(t_1,t_2) = \frac{n!}{(i-1)!(j-i-1)!(n-j)!}\sum_{\ell=0}^{\infty}\frac{(t_1+\ell-1)^{(\ell)}}{\ell!}B(t_1+i+\ell,j-i)$$

$$\int_0^1 v^{j+t_1+t_2+\ell-1}(1-v)^{n-j-t_2}\,dv$$

$$= \frac{n!}{(i-1)!(j-i-1)!(n-j)!}\sum_{\ell=0}^{\infty}\frac{(t_1+\ell-1)^{(\ell)}}{\ell!}B(t_1+i+\ell,j-i)$$

$$B(j+t_1+t_2+\ell,n-j-t_2+1)$$

$$= \frac{\Gamma(n+1)}{\Gamma(i)\Gamma(n-j+1)}\sum_{\ell=0}^{\infty}\frac{(t_1+\ell-1)^{(\ell)}}{\ell!}\frac{\Gamma(t_1+i+\ell)}{\Gamma(t_1+j+\ell)}$$

$$\frac{\Gamma(t_1+t_2+j+\ell)\,\Gamma(n-j+1-t_2)}{\Gamma(n+1+t_1+\ell)}. \tag{3.13}$$

From the above expression for the joint moment generating function of $Z_{i:n}$ and $Z_{j:n}$, one can obtain the product moments as follows:

$$\alpha^{*(k_1,k_2)}_{i,j:n} = E(Z^{k_1}_{i:n}\,Z^{k_2}_{j:n}) = \frac{\partial^{k_1+k_2}}{\partial t_1^{k_1}\,\partial t_2^{k_2}} M^{*}_{i,j:n}(t_1,t_2)\Big|_{t_1=t_2=0}. \tag{3.14}$$

The case $k_1=k_2=1$ is of particular importance and, in this case, we get

$$\alpha^*_{i,j:n} = \psi'(j) + \left\{\psi(i) - \psi(n+1)\right\} \left\{\psi(j) - \psi(n-j+1)\right\}$$

$$+ \sum_{\ell=1}^{\infty} \frac{1}{\ell} \frac{(i+\ell-1)^{(\ell)}}{(n+\ell)^{(\ell)}} \left\{\psi(j+\ell) - \psi(n-j+1)\right\} . \tag{3.15}$$

Shah (1966) tabulated the covariances of order statistics for sample sizes up to ten while Gupta, Qureishi and Shah (1967) extended up to 25. Recently, Balakrishnan and Malik (1990) provided tables of means, variances and covariances for sample sizes 50 and less. It should also be mentioned here that Balakrishnan and Leung (1988 a,b) derived series expressions similar to the ones given in Eqs. (3.2)–(3.4) and (3.15) for the single and the product moments of order statistics from a generalized logistic distribution and provided tables of means, variances and covariances for sample sizes up to 15.

4. RECURRENCE RELATIONS FOR MOMENTS

In this section we shall present some recurrence relations satisfied by the single and the product moments that were established by Shah (1966, 1970) and show that one may evaluate these moments for all order statistics from all sample sizes in a simple and systematic recursive manner.

Relation 4.1: For $n \geq 1$ and $k=0,1,2,\ldots$,

$$\alpha^{*(k+1)}_{1:n+1} = \alpha^{*(k+1)}_{1:n} - \frac{k+1}{n} \alpha^{*(k)}_{1:n} \tag{4.1}$$

with $\alpha^{*(0)}_{1:n} \equiv 1$ for $n=1,2,\ldots.$.

Proof. From (1.3) we have for $n \geq 1$ and $k \geq 0$

$$\alpha_{1:n}^{*(k)} = n \int_{-\infty}^{\infty} z^k \left\{1-F^*(z)\right\}^{n-1} f^*(z)\, dz. \tag{4.2}$$

By using the relation (1.5) in Eq. (4.2), we get

$$\alpha_{1:n}^{*(k)} = n \int_{-\infty}^{\infty} z^k\, F^*(z) \left\{1-F^*(z)\right\}^{n} dz$$

which, upon integrating by parts, yields

$$\alpha_{1:n}^{*(k)} = \frac{n}{k+1} \left[n \int_{-\infty}^{\infty} z^{k+1}\, F^*(z) \left\{1-F^*(z)\right\}^{n-1} f^*(z)\, dz \right.$$

$$\left. - \int_{-\infty}^{\infty} z^{k+1} \left\{1-F^*(z)\right\}^{n} f^*(z)\, dz \right]$$

$$= \frac{n}{k+1} \left\{ \alpha_{1:n}^{*(k+1)} - \alpha_{1:n+1}^{*(k+1)} \right\}.$$

The recurrence relation in (4.1) is obtained by rewriting the above equation.

Relation 4.2: For $1 \leq i \leq n$ and $k=0,1,2,\ldots\ldots$,

$$\alpha_{i+1:n+1}^{*(k+1)} = \alpha_{i:n+1}^{*(k+1)} + \frac{(k+1)(n+1)}{i(n-i+1)}\, \alpha_{i:n}^{*(k)}. \tag{4.3}$$

Proof. From (1.3) we have for $1 \le i \le n$ and $k \ge 0$

$$\alpha_{i:n}^{*(k)} = \frac{n!}{(i-1)!(n-i)!} \int_{-\infty}^{\infty} z^k \left\{F^*(z)\right\}^{i-1} \left\{1-F^*(z)\right\}^{n-i} f^*(z)dz \,. \tag{4.4}$$

By using the relation (1.5) in Eq. (4.4), we get

$$\alpha_{i:n}^{*(k)} = \frac{n!}{(i-1)!(n-i)!} \int_{-\infty}^{\infty} z^k \left\{F^*(z)\right\}^{i} \left\{1-F^*(z)\right\}^{n-i+1} dz$$

which, upon integrating by parts, yields

$$\alpha_{i:n}^{*(k)} = \frac{n!}{(i-1)!(n-i)!(k+1)} \left[(n-i+1) \int_{-\infty}^{\infty} z^{k+1} \left\{F^*(z)\right\}^{i} \left\{1-F^*(z)\right\}^{n-i} f^*(z)dz \right.$$

$$\left. - \, i \int_{-\infty}^{\infty} z^{k+1} \left\{F^*(z)\right\}^{i-1} \left\{1-F^*(z)\right\}^{n-i+1} f^*(z)dz\right]$$

$$= \frac{i\,(n-i+1)}{(k+1)(n+1)} \left\{\alpha_{i+1:n+1}^{*(k+1)} - \alpha_{i:n+1}^{*(k+1)}\right\} .$$

The recurrence relation in (4.3) follows by rewriting the above equation.

With the values of $\alpha_{1:1}^{*(j)}$ $(j=1,2,\ldots,k)$ known, one may be able to use Relations 4.1 and 4.2 in a simple recursive way to compute the first k single moments of all order statistics from all sample sizes. Thus, for example, by starting with the values of $\alpha_{1:1}^{*(1)}=0$ and $\alpha_{1:1}^{*(2)}=\pi^2/3$, one may employ Relations 4.1 and 4.2 to evaluate the first

two single moments and, thence, the variances of all order statistics from all sample sizes in a simple and systematic recursive process. These computations may be checked by using the identities (David, 1981, p. 39; Arnold and Balakrishnan, 1989, p. 6)

$$\sum_{i=1}^{n} \alpha_{i:n}^{*(j)} = n\alpha_{1:1}^{*(j)},\ j=1,2,\ldots; \tag{4.5}$$

see also Balakrishnan and Malik (1986).

Relation 4.3: For $1 \leq i \leq n-1$,

$$\alpha^{*}_{i,i+1:n+1} = \alpha_{i:n+1}^{*(2)} + \frac{n+1}{n-i+1}\left\{\alpha^{*}_{i,i+1:n} - \alpha_{i:n}^{*(2)} - \frac{1}{n-i}\,\alpha^{*}_{i:n}\right\}. \tag{4.6}$$

Proof. For $1 \leq i \leq n-1$, we may write from (1.4) that

$$\alpha^{*}_{i:n} = E(Z_{i:n}\, Z^{0}_{i+1:n})$$

$$= \frac{n!}{(i-1)!(n-i-1)!}\int_{-\infty}^{\infty} z_1 \left\{F^{*}(z_1)\right\}^{i-1} f^{*}(z_1)\, K(z_1)\, dz_1, \tag{4.7}$$

where

$$K(z_1) = \int_{z_1}^{\infty} \left\{1-F^{*}(z_2)\right\}^{n-i-1} f^{*}(z_2)\, dz_2\ . \tag{4.8}$$

By using the relation (1.5) in Eq. (4.8) and integrating by parts, we get

$$K(z_1) = -z_1\left\{1-F^*(z_1)\right\}^{n-i} + z_1\left\{1-F^*(z_1)\right\}^{n-i+1}$$

$$+ (n-i)\int_{z_1}^{\infty} z_2 \left\{1-F^*(z_2)\right\}^{n-i-1} f^*(z_2)\, dz_2$$

$$-(n-i+1)\int_{z_1}^{\infty} z_2 \left\{1-F^*(z_2)\right\}^{n-i} f^*(z_2)\, dz_2 \ .$$

Upon substituting the above expression of $K(z_1)$ in (4.7) and simplifying the resulting equation, we get

$$\alpha^*_{i:n} = -(n-i)\alpha^{*(2)}_{i:n} + \frac{(n-i)(n-i+1)}{(n+1)}\alpha^{*(2)}_{i:n+1}$$

$$+ (n-i)\alpha^*_{i,i+1:n} - \frac{(n-i)(n-i+1)}{(n+1)}\alpha^*_{i,i+1:n+1}.$$

The recurrence relation in (4.6) follows by rewriting the above equation.

Relation 4.4: For $1 \le i \le n-1$,

$$\alpha^*_{i+1,i+2:n+1} = \alpha^{*(2)}_{i+2:n+1} + \frac{n+1}{i+1}\left\{\frac{1}{i}\alpha^*_{i+1:n} + \alpha^*_{i,i+1:n} - \alpha^{*(2)}_{i+1:n}\right\}. \quad (4.9)$$

Proof. For $1 \leq i \leq n-1$, we may write from (1.4) that

$$\alpha^*_{i+1:n} = E(Z^0_{i:n}\, Z_{i+1:n})$$

$$= \frac{n!}{(i-1)!(n-i-1)!} \int_{-\infty}^{\infty} z_2 \left\{1-F^*(z_2)\right\}^{n-i-1} f^*(z_2)\, K(z_2)\, dz_2, \tag{4.10}$$

where

$$K(z_2) = \int_{-\infty}^{z_2} \left\{F^*(z_1)\right\}^{i-1} f^*(z_1)\, dz_1 \ . \tag{4.11}$$

By using the relation (1.5) in Eq. (4.11) and integrating by parts, we get

$$K(z_2) = z_2\left\{F^*(z_2)\right\}^{i} - z_2\left\{F^*(z_2)\right\}^{i+1}$$

$$- i \int_{-\infty}^{z_2} z_1 \left\{F^*(z_1)\right\}^{i-1} f^*(z_1)\, dz_1$$

$$+ (i+1) \int_{-\infty}^{z_2} z_1 \left\{F^*(z_1)\right\}^{i} f^*(z_1)\, dz_1 \ .$$

Upon substituting the above expression of $K(z_2)$ in (4.10) and simplifying the resulting equation, we get

$$\alpha^*_{i+1:n} = i\, \alpha^{*(2)}_{i+1:n} - \frac{i(i+1)}{(n+1)}\, \alpha^{*(2)}_{i+2:n+1}$$

$$- i\, \alpha^*_{i,i+1:n} + \frac{i(i+1)}{(n+1)}\, \alpha^*_{i+1,i+2:n+1}.$$

The recurrence relation in (4.9) is obtained by rewriting the above equation.

In particular, by setting i=n–1 in Relation 4.4 we get for $n \geq 2$

$$\alpha^*_{n,n+1:n+1} = \alpha^{*(2)}_{n+1:n+1} + \frac{n+1}{n}\left\{\frac{1}{n-1}\alpha^*_{n:n} + \alpha^*_{n-1,n:n} - \alpha^{*(2)}_{n:n}\right\}. \qquad (4.12)$$

It should be mentioned here that the recurrence relations in (4.6) and (4.12) are sufficient for the evaluation of all the product moments of order statistics from all sample sizes. By starting with the result that $\alpha^*_{1,2:2}=\alpha^{*2}_{1:1}=0$ (Govindarajulu, 1963; Joshi, 1971), the recurrence relations in (4.6) and (4.12) will enable one to compute all the immediate upper–diagonal product moments $\alpha^*_{i,i+1:n}$ $(1 \leq i \leq n-1)$ for all sample sizes in a simple recursive way. All the remaining product moments, viz., $\alpha^*_{i,j:n}$ for $1 \leq i < j \leq n$ and $j-i \geq 2$, may be determined systematically by employing the well–known recurrence relation (David, 1981, p. 48; Arnold and Balakrishnan, 1989, p.10)

$$(i-1)\alpha^*_{i,j:n} + (j-i)\alpha^*_{i-1,j:n} + (n-j+1)\alpha^*_{i-1,j-1:n} = n\alpha^*_{i-1,j-1:n-1} \qquad (4.13)$$

that is true for any arbitrary distribution. These computations may then be checked by using the identity (David, 1981, p. 39; Arnold and Balakrishnan, 1989, p. 10)

$$\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \alpha^*_{i,j:n} = \binom{n}{2}\alpha^{*2}_{1:1}; \qquad (4.14)$$

see also Balakrishnan and Malik (1986).

By proceeding on similar lines, we may also establish the following recurrence relations.

Relation 4.5: For $1 \le i < j \le n$ and $j-i \ge 2$,

$$\alpha^*_{i,j:n+1} = \alpha^*_{i,j-1:n+1} + \frac{n+1}{n-j+2}\left\{\alpha^*_{i,j:n} - \alpha^*_{i,j-1:n} - \frac{1}{n-j+1}\,\alpha^*_{i:n}\right\}. \tag{4.15}$$

Relation 4.6: For $1 \le i < j \le n$ and $j-i \ge 2$,

$$\alpha^*_{i+1,j+1:n+1} = \alpha^*_{i+2,j+1:n+1} + \frac{n+1}{i+1}\left\{\frac{1}{i}\,\alpha^*_{j:n} + \alpha^*_{i,j:n} - \alpha^*_{i+1,j:n}\right\}. \tag{4.16}$$

It should be pointed out here that one may employ Relations 4.5 and 4.6 to determine all the product moments other than the immediate upper–diagonal product moments, viz., $\alpha^*_{i,j:n}$ for $1 \le i < j \le n$ and $j-i \ge 2$, instead of the recurrence relation in (4.13).

5. DISTRIBUTIONS OF SOME SYSTEMATIC STATISTICS

In this section we first present the distribution of the sample range as derived by Gupta and Shah (1965). Then, we give the expression of the distribution of the r^{th} quasi–range derived by Malik (1980).

Let us denote the sample range $Z_{n:n} - Z_{1:n}$ by W_n. The cumulative distribution function of W_n can be written down as (David, 1981, p. 12)

$$\Pr(W_n \le w) = n \int_{-\infty}^{\infty} \left\{F^*(z+w) - F^*(z)\right\}^{n-1} f^*(z)\, dz, \quad 0 \le w < \infty. \tag{5.1}$$

Expanding the term $\left\{F^*(z+w) - F^*(z)\right\}^{n-1}$ binomially, we get from (5.1) that

$$\Pr(W_n \le w) = n \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \int_{-\infty}^{\infty} \left\{F^*(z+w)\right\}^{n-1-k} \left\{F^*(z)\right\}^k f^*(z)\, dz$$

$$= n \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \int_{-\infty}^{\infty} \frac{e^{-z}}{(1+e^{-w}e^{-z})^{n-1-k}\ (1+e^{-z})^{k+2}}\, dz. \qquad (5.2)$$

By substituting $u=1/(1+e^{-w}e^{-z})$ in the integral in (5.2), we get

$$\Pr(W_n \le w) = n \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} e^{-(k+1)w} A_{k,n}(w), \quad 0 \le w < \infty, \qquad (5.3)$$

where, with $a = e^{-w}-1$,

$$A_{k,n}(w) = \int_0^1 u^{n-1} (1+au)^{-k-2}\, du$$

$$= \frac{1}{(-a)^n} \left[(-1)^{k+1} \binom{n-1}{k+1} \ln(1+a) + \sum_{\substack{\ell=0 \\ \ell \ne k+1}}^{n-1} (-1)^\ell \binom{n-1}{\ell} \frac{1}{(\ell-k-1)}\left\{(1+a)^{\ell-k-1}-1\right\}\right]$$

$$= \frac{1}{(1-e^{-w})^n} \left[(-1)^k \binom{n-1}{k+1} w + \sum_{\substack{\ell=0 \\ \ell \ne k+1}}^{n-1} (-1)^\ell \binom{n-1}{\ell} \frac{1}{(\ell-k-1)}\left\{e^{-(\ell-k-1)w}-1\right\}\right].$$

(5.4)

In the expression of $A_{k,n}(w)$ in (5.4), $\binom{n-1}{k+1}$ should be set to zero if $k>n-2$. By substituting the expression of $A_{k,n}(w)$ in (5.4) into Eq. (5.3), we derive the cumulative distribution function of the sample range W_n as

$$\Pr(W_n \leq w) = \frac{n}{(1-e^{-w})^n} \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \Bigg[(-1)^k \binom{n-1}{k+1} w\, e^{-(k+1)w}$$

$$+ \sum_{\substack{\ell=0 \\ \ell \neq k+1}}^{n-1} (-1)^\ell \binom{n-1}{\ell} \frac{1}{(\ell-k-1)} \left\{ e^{-\ell w} - e^{-(k+1)w} \right\} \Bigg], \quad 0 \leq w < \infty. \tag{5.5}$$

In particular, for n=2 and 3, we obtain from (5.5) that

$$\Pr(W_2 \leq w) = \left\{ 1 - e^{-2w} - 2we^{-w} \right\} / (1-e^{-w})^2 \tag{5.6}$$

and

$$\Pr(W_3 \leq w) = \left\{ 1+9e^{-w}-9e^{-2w}-e^{-3w}-6we^{-w}(1+e^{-w}) \right\} / (1-e^{-w})^3. \tag{5.7}$$

Gupta and Shah (1965) tabulated the probability integrals of the range for n=2 and 3 from (5.6) and (5.7), repsectively.

By differentiating the distribution function of the sample range in (5.5) with respect to w, we derive the density function of the sample range W_n as

$$f_{W_n}(w) = -\frac{n^2 e^{-w}}{(1-e^{-w})^{n+1}} \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \Bigg[(-1)^k \binom{n-1}{k+1} w e^{-(k+1)w}$$

$$+ \sum_{\substack{\ell=0 \\ \ell \neq k+1}}^{n-1} (-1)^\ell \binom{n-1}{\ell} \frac{1}{(\ell-k-1)} \left\{ e^{-\ell w} - e^{-(k+1)w} \right\} \Bigg]$$

$$+ \frac{n}{(1-e^{-w})^n} \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \Bigg[(-1)^k \binom{n-1}{k+1} e^{-(k+1)w} \left\{ 1-(k+1)w \right\}$$

$$- \sum_{\substack{\ell=0 \\ \ell \neq k+1}}^{n-1} (-1)^\ell \binom{n-1}{\ell} \frac{1}{(\ell-k-1)} \left\{ \ell e^{-\ell w} - (k+1) e^{-(k+1)w} \right\} \Bigg], \quad 0 \le w < \infty, \tag{5.8}$$

where, as before, $\binom{n-1}{k+1}$ should be set to zero if $k > n-2$.

Proceeding exactly on similar lines, Malik (1980) derived the cumulative distribution function of the r^{th} quasi-range $W_{n,r} = Z_{n-r:n} - Z_{r+1:n}$ $(r=0,1,\ldots,[\frac{n-1}{2}])$ to be

$$\Pr(W_{n,r} \le w) = \sum_{k=0}^{r} \frac{\prod_{i=0}^{2r-k} (n-i)}{r!\,(r-k)!} \Bigg[\sum_{j=0}^{n-2r+k-1} (-1)^j \begin{bmatrix} n-2r+k-1 \\ j \end{bmatrix} e^{-(r+j+1)w}$$

$$\Bigg\{ \sum_{\ell=0}^{r-k} \frac{(-1)^{\ell+1} \begin{bmatrix} r-k \\ \ell \end{bmatrix}}{(1-e^{-w})^{n-r+k+\ell}} \Bigg[(-1)^{r+j} \begin{bmatrix} n-r+k+\ell-1 \\ r+j+1 \end{bmatrix} w$$

$$+ \sum_{\substack{m=0 \\ m \neq r+j+1}}^{n-r+k+\ell-1} (-1)^m \begin{bmatrix} n-r+k+\ell-1 \\ m \end{bmatrix} \frac{e^{-(m-r-j-1)w} - 1}{m-r-j-1} \Bigg] \Bigg\} \Bigg],$$

$$0 \le w < \infty, \tag{5.9}$$

where $\begin{bmatrix} n-r+k+\ell-1 \\ r+j+1 \end{bmatrix}$ should be set to zero if $j>n-2r+k+\ell-2$. The distribution function of the sample range in (5.8) may be derived as a special case from (5.9) by setting $r=0$.

6. RESULTS FOR TRUNCATED DISTRIBUTIONS

In this section we start with order statistics from a doubly truncated logistic distribution and present the results of Balakrishnan and Joshi (1983 a) and Balakrishnan and Kocherlakota (1986). In addition to generalizing the relations given in Section 4, these results will enable one to evaluate the single and the product moments of all order statistics from all sample sizes in a simple recursive manner.

Let $Z^*_{1:n} \le Z^*_{2:n} \le \ldots\ldots \le Z^*_{n:n}$ be order statistics from a random sample of size n from a doubly truncated logistic distribution with probability density function

$$f^{**}(z) = \begin{cases} \dfrac{1}{(P-Q)} \dfrac{e^{-z}}{(1+e^{-z})^2} & , Q \le z \le P_1 \\ 0 & , \text{otherwise} \end{cases} \tag{6.1}$$

and cumulative distribution function

$$F^{**}(z) = \frac{1}{(P-Q)} \left\{ \frac{1}{(1+e^{-z})} - Q \right\}, \quad Q_1 \le z \le P_1, \tag{6.2}$$

where Q and 1–P are the proportions of truncation on the left and the right of the standard logistic density function in (1.1). Under this notation,

$$Q_1 = \ln\left[\frac{Q}{1-Q}\right], \qquad P_1 = \ln\left[\frac{P}{1-P}\right], \tag{6.3}$$

and let

$$Q_2 = \frac{Q(1-Q)}{P-Q}, \qquad P_2 = \frac{P(1-P)}{P-Q}. \tag{6.4}$$

From (6.1) and (6.2) we observe the relations

$$f^{**}(z) = (1-2Q)F^{**}(z) - (P-Q)\{F^{**}(z)\}^2 + Q_2, \tag{6.5}$$

$$f^{**}(z) = (2P-1)\{1-F^{**}(z)\} - (P-Q)\{1-F^{**}(z)\}^2 + P_2, \tag{6.6}$$

and

$$f^{**}(z) = (2P-1)F^{**}(z)\{1-F^{**}(z)\} + (P+Q-1)\{1-F^{**}(z)\}^2 + P_2. \tag{6.7}$$

Let us now denote the single moments $E(Z^{*k}_{i:n})$ by $\alpha^{**(k)}_{i:n}$ $(1 \le i \le n, k \ge 1)$ and the product moments $E(Z^{*}_{i:n}Z^{*}_{j:n})$ by $\alpha^{**}_{i,j:n}$ $(1 \le i < j \le n)$. For convenience, let us also use $\alpha^{**}_{i:n}$ for $\alpha^{**(1)}_{i:n}$ and $\alpha^{**}_{i,i:n}$ for $\alpha^{**(2)}_{i:n}$. Then, these moments satisfy the following recurrence relations.

<u>Relation 6.1</u>: For k=0,1,2,...,

$$\alpha^{**(k+1)}_{1:2} = Q_1^{k+1} + \frac{1}{P-Q}\Big[P_2\Big\{P_1^{k+1} - Q_1^{k+1}\Big\} + (2P-1)\Big\{\alpha^{**(k+1)}_{1:1} - Q_1^{k+1}\Big\} - (k+1)\alpha^{**(k)}_{1:1}\Big]. \tag{6.8}$$

Proof. For k≥0, let us consider

$$\alpha_{1:1}^{**(k)} = \int_{Q_1}^{P_1} z^k f^{**}(z)\, dz .$$

Upon using (6.6) in the above equation and then integrating by parts, we get

$$\alpha_{1:1}^{**(k)} = \frac{1}{k+1}\Big[(2P-1)\Big\{\alpha_{1:1}^{**(k+1)} - Q_1^{k+1}\Big\} - (P-Q)\Big\{\alpha_{1:2}^{**(k+1)} - Q_1^{k+1}\Big\}$$

$$+ P_2\Big\{P_1^{k+1} - Q_1^{k+1}\Big\}\Big] . \qquad (6.9)$$

The recurrence relation in (6.8) follows by rewriting Eq. (6.9).

Relation 6.2: For k=0,1,2,....,

$$\alpha_{2:2}^{**(k+1)} = P_1^{k+1} - \frac{1}{P-Q}\Big[Q_2\Big\{P_1^{k+1} - Q_1^{k+1}\Big\} + (1-2Q)\Big\{P_1^{k+1} - \alpha_{1:1}^{**(k+1)}\Big\}$$

$$- (k+1)\alpha_{1:1}^{**(k)}\Big] . \qquad (6.10)$$

Proof. For k≥0, let us consider

$$\alpha_{1:1}^{**(k)} = \int_{Q_1}^{P_1} z^k f^{**}(z)\, dz .$$

Upon using (6.5) in the above equation and then integrating by parts, we get

$$\alpha_{1:1}^{**(k)} = \frac{1}{k+1}\Big[Q_2\Big\{P_1^{k+1} - Q_1^{k+1}\Big\} + (1-2Q)\Big\{P_1^{k+1} - \alpha_{1:1}^{**(k+1)}\Big\}$$
$$- (P-Q)\Big\{P_1^{k+1} - \alpha_{2:2}^{**(k+1)}\Big\}\Big] . \tag{6.11}$$

The recurrence relation in (6.10) is obtained by rewriting Eq. (6.11).

<u>Relation 6.3</u>: For $n \geq 2$ and $k=0,1,2,\ldots,$

$$\alpha_{1:n+1}^{**(k+1)} = Q_1^{k+1} + \frac{1}{P-Q}\Big[P_2\Big\{\alpha_{1:n-1}^{**(k+1)} - Q_1^{k+1}\Big\} + (2P-1)\Big\{\alpha_{1:n}^{**(k+1)} - Q_1^{k+1}\Big\}$$
$$- \frac{k+1}{n}\alpha_{1:1}^{**(k)}\Big] . \tag{6.12}$$

<u>Proof</u>. For $n \geq 2$ and $k \geq 0$, let us consider

$$\alpha_{1:n}^{**(k)} = n\int_{Q_1}^{P_1} z^k \Big\{1 - F^{**}(z)\Big\}^{n-1} f^{**}(z)\, dz .$$

Upon using (6.6) in the above equation and then integrating by parts, we get

$$\alpha_{1:n}^{**(k)} = \frac{n}{k+1}\Big[P_2\Big\{\alpha_{1:n-1}^{**(k+1)} - Q_1^{k+1}\Big\} + (2P-1)\Big\{\alpha_{1:n}^{**(k+1)} - Q_1^{k+1}\Big\}$$
$$- (P-Q)\Big\{\alpha_{1:n+1}^{**(k+1)} - Q_1^{k+1}\Big\}\Big] . \tag{6.13}$$

The recurrence relation in (6.12) follows by rewriting Eq. (6.13).

Relation 6.4: For k=0,1,2,....,

$$\alpha_{2:3}^{**(k+1)}=\alpha_{1:3}^{**(k+1)}+\frac{3}{P-Q}\Big[P_2\Big\{P_1^{k+1}-\alpha_{1:1}^{**(k+1)}\Big\}+\frac{2P-1}{2}\Big\{\alpha_{2:2}^{**(k+1)}-\alpha_{1:2}^{**(k+1)}\Big\}$$

$$-\frac{k+1}{2}\,\alpha_{2:2}^{**(k)}\Big]\,. \qquad (6.14)$$

Proof. For $k\geq 0$, let us consider

$$\alpha_{2:2}^{**(k)}=2\int_{Q_1}^{P_1} z^k\,F^{**}(z)\,f^{**}(z)\,dz\,.$$

Upon using (6.6) in the above equation and then integrating by parts, we get

$$\alpha_{2:2}^{**(k)}=\frac{1}{k+1}\Big[(2P-1)\Big\{\alpha_{2:2}^{**(k+1)}-\alpha_{1:2}^{**(k+1)}\Big\}-\frac{2}{3}(P-Q)\Big\{\alpha_{2:3}^{**(k+1)}-\alpha_{1:3}^{**(k+1)}\Big\}$$

$$+\,2P_2\Big\{P_1^{k+1}-\alpha_{1:1}^{**(k+1)}\Big\}\Big]\,. \qquad (6.15)$$

The recurrence relation in (6.14) is obtained by rewriting Eq. (6.15).

Relation 6.5: For $n\geq 3$ and k=0,1,2,....,

$$\alpha_{2:n+1}^{**(k+1)}=\alpha_{1:n+1}^{**(k+1)}+\frac{n+1}{P-Q}\Big[\frac{P_2}{n-1}\Big\{\alpha_{2:n-1}^{**(k+1)}-\alpha_{1:n-1}^{**(k+1)}\Big\}+\frac{2P-1}{n}$$

$$\Big\{\alpha_{2:n}^{**(k+1)}-\alpha_{1:n}^{**(k+1)}\Big\}-\frac{k+1}{n(n-1)}\,\alpha_{2:n}^{**(k)}\Big]\,. \qquad (6.16)$$

Proof. For $n \geq 3$ and $k \geq 0$, let us consider

$$\alpha_{2:n}^{**(k)} = n(n-1)\int_{Q_1}^{P_1} z^k\, F^{**}(z)\,\{1-F^{**}(z)\}^{n-2}\, f^{**}(z)\,dz\,.$$

Upon using (6.6) in the above equation and then integrating by parts, we get

$$\alpha_{2:n}^{**(k)} = \frac{n-1}{k+1}\Big[(2P-1)\Big\{\alpha_{2:n}^{**(k+1)} - \alpha_{1:n}^{**(k+1)}\Big\} - (P-Q)\frac{n}{n+1}\Big\{\alpha_{2:n+1}^{**(k+1)} - \alpha_{1:n+1}^{**(k+1)}\Big\}$$
$$+ P_2\Big\{\alpha_{2:n}^{**(k+1)} - \alpha_{1:n}^{**(k+1)}\Big\}\Big]\,. \qquad (6.17)$$

The recurrence relation in (6.16) follows by rewriting Eq. (6.17).

Relation 6.6: For $2 \leq i \leq n-1$ and $k=0,1,2,\ldots,$

$$\alpha_{i+1:n+1}^{**(k+1)} = \frac{n+1}{i(2P-1)}\Big[\frac{k+1}{n-i+1}\,\alpha_{i:n}^{**(k)} - \frac{nP_2}{n-i+1}\Big\{\alpha_{i:n-1}^{**(k+1)} - \alpha_{i-1:n-1}^{**(k+1)}\Big\}$$
$$- \frac{1}{n+1}\Big\{(n+1)(P+Q-1) - i(2P-1)\Big\}\,\alpha_{i:n+1}^{**(k+1)}$$
$$+ (P+Q-1)\alpha_{i-1:n}^{**(k+1)}\Big]\,. \qquad (6.18)$$

Proof. For $2 \le i \le n-1$ and $k \ge 0$, let us consider

$$\alpha_{i:n}^{**(k)} = \frac{n!}{(i-1)!(n-i)!} \int_{Q_1}^{P_1} z^k \left\{F^{**}(z)\right\}^{i-1} \left\{1-F^{**}(z)\right\}^{n-i} f^{**}(z)\, dz\,.$$

Upon using (6.7) in the above equation and then integrating by parts, we get

$$\begin{aligned}\alpha_{i:n}^{**(k)} = \frac{1}{k+1} \Bigg[& \frac{i(n-i+1)}{n+1} (2P-1) \left\{\alpha_{i+1:n+1}^{**(k+1)} - \alpha_{i:n+1}^{**(k+1)}\right\} \\ & - \frac{(n-i+1)(n-i+2)}{n+1} (P+Q-1) \left\{\alpha_{i:n+1}^{**(k+1)} - \alpha_{i-1:n-1}^{**(k+1)}\right\} \\ & + nP_2 \left\{\alpha_{i:n-1}^{**(k+1)} - \alpha_{i-1:n-1}^{**(k+1)}\right\}\Bigg]\,. \qquad (6.19)\end{aligned}$$

If we now use the well-known relation (David, 1981, p. 46; Arnold and Balakrishnan, 1989, p. 6)

$$(n-i+2)\left\{\alpha_{i:n+1}^{**(k+1)} - \alpha_{i-1:n-1}^{**(k+1)}\right\} = (n+1)\left\{\alpha_{i:n+1}^{**(k+1)} - \alpha_{i-1:n}^{**(k+1)}\right\}$$

in (6.19) and simplify the resulting equation, we derive the recurrence relation in (6.18).

Relation 6.7: For $n \ge 2$ and $k=0,1,2,....,$

$$\begin{aligned}\alpha_{n+1:n+1}^{**(k+1)} = \frac{n+1}{n(2P-1)} \Bigg[& (k+1)\alpha_{n:n}^{**(k)} - nP_2\left\{P_1^{k+1} - \alpha_{n-1:n-1}^{**(k+1)}\right\} \\ & - \frac{1}{n+1}\left\{(n+1)(P+Q-1) - n(2P-1)\right\} \alpha_{n:n+1}^{**(k+1)} \\ & + (P+Q-1)\alpha_{n-1:n}^{**(k+1)}\Bigg]\,. \qquad (6.20)\end{aligned}$$

Proof. For $n \geq 2$ and $k \geq 0$, let us consider

$$\alpha_{n:n}^{**(k)} = n \int_{Q_1}^{P_1} z^k \left\{F^{**}(z)\right\}^{n-1} f^{**}(z)\, dz .$$

Upon using (6.7) in the above equation and then integrating by parts, we get

$$\begin{aligned}\alpha_{n:n}^{**(k)} = \frac{1}{k+1} \Bigg[& \frac{n}{n+1} (2P-1) \left\{\alpha_{n+1:n+1}^{**(k+1)} - \alpha_{n:n+1}^{**(k+1)}\right\} \\ & + \frac{2}{n+1}(P+Q-1)\left\{\alpha_{n:n+1}^{**(k+1)} - \alpha_{n-1:n-1}^{**(k+1)}\right\} \\ & + nP_2\left\{P_1^{k+1} - \alpha_{n-1:n-1}^{**(k+1)}\right\}\Bigg] . \qquad (6.21)\end{aligned}$$

If we now use the well–known relation (David, 1981, p. 46; Arnold and Balakrishnan, 1989, p. 6)

$$2\left\{\alpha_{n:n+1}^{**(k+1)} - \alpha_{n-1:n-1}^{**(k+1)}\right\} = (n+1)\left\{\alpha_{n:n+1}^{**(k+1)} - \alpha_{n-1:n}^{**(k+1)}\right\}$$

in (6.21) and simplify the resulting equation, we derive the recurrence relation in (6.20).

By starting with the values of $\alpha_{1:1}^{**(j)}$ ($j=1,2,\ldots,k$), one can employ Relations 6.1 – 6.7 in a simple and systematic recursive way to compute the first k single moments of all order statistics from all sample sizes for any choice of Q and P.

We may also establish the following recurrence relations satisfied by the product moments of order statistics.

Relation 6.8: We have

$$\alpha_{1,2:3}^{**} = \alpha_{1:3}^{**(2)} + \frac{3}{2(P-Q)}\Big[2P_2\Big\{P_1\alpha_{1:1}^{**(1)} - \alpha_{1:1}^{**(2)}\Big\}$$

$$+ (2P-1)\Big\{\alpha_{1,2:2}^{**} - \alpha_{1:2}^{**(2)}\Big\} - \alpha_{1:2}^{**(1)}\Big]\,. \tag{6.22}$$

Proof. Let us start with

$$\alpha_{1:2}^{**(1)} = E(Z_{1:2}^{*}\, Z_{2:2}^{*0})$$

$$= 2\int_{Q_1}^{P_1} z_1\, f^{**}(z_1)\, K(z_1)\, dz_1, \tag{6.23}$$

where

$$K(z_1) = \int_{z_1}^{P_1} f^{**}(z_2)\, dz_2\,. \tag{6.24}$$

Upon using (6.6) in (6.24) and integrating by parts, we get

$$K(z_1) = (2P-1)\left\{\int_{z_1}^{P_1} z_2\, f^{**}(z_2)\, dz_2 - z_1\left\{1 - F^{**}(z_1)\right\}\right\}$$

$$-\,(P-Q)\left\{2\int_{z_1}^{P_1} z_2\left\{1-F^{**}(z_2)\right\} f^{**}(z_2)\, dz_2 - z_1\left\{1-F^{**}(z_1)\right\}^2\right\}$$

$$+\, P_2\,(P_1 - z_1).$$

Upon substituting the above expression of $K(z_1)$ in (6.23) and simplifying the resulting equation, we derive the recurrence relation in (6.22).

<u>Relation 6.9</u>: We have

$$\alpha^{**}_{2,3:3} = \alpha^{**(2)}_{3:3} + \frac{3}{2(P-Q)}\Big[\alpha^{**(1)}_{2:2} - 2Q_2\Big\{\alpha^{**(2)}_{1:1} - Q_1\alpha^{**(1)}_{1:1}\Big\}$$

$$- (1-2Q)\Big\{\alpha^{**(2)}_{2:2} - \alpha^{**}_{1,2:2}\Big\}\Big] . \qquad (6.25)$$

<u>Proof</u>. Let us start with

$$\alpha^{**(1)}_{2:2} = E(Z^{*0}_{1:2}\, Z^{*}_{2:2})$$

$$= 2\int_{Q_1}^{P_1} z_2\, f^{**}(z_2)\, K(z_2)\, dz_2, \qquad (6.26)$$

where

$$K(z_2) = \int_{Q_1}^{z_2} f^{**}(z_1)\, dz_1 . \qquad (6.27)$$

Upon using (6.5) in (6.27) and integrating by parts, we get

$$K(z_2) = (1-2Q)\Big\{z_2\, F^{**}(z_2) - \int_{Q_1}^{z_2} z_1\, f^{**}(z_1)\, dz_1\Big\}$$

$$- (P-Q)\Big[z_2\Big\{F^{**}(z_2)\Big\}^2 - 2\int_{Q_1}^{z_2} z_1\, F^{**}(z_1)\, f^{**}(z_1)\, dz_1\Big]$$

$$+ Q_2\,(z_2-Q_1).$$

Upon substituting the above expression of $K(z_2)$ in (6.26) and simplifying the resulting equation, we derive the recurrence relation in (6.25).

Relation 6.10: For $1 \le i \le n-2$,

$$\alpha^{**}_{i,i+1:n+1} = \alpha^{**(2)}_{i:n+1} + \frac{(n+1)}{(n-i+1)(P-Q)}\left[\frac{n}{n-i} P_2\left\{\alpha^{**}_{i,i+1:n-1} - \alpha^{**(2)}_{i:n-1}\right\}\right.$$
$$\left. + (2P-1)\left\{\alpha^{**}_{i,i+1:n} - \alpha^{**(2)}_{i:n}\right\} - \frac{1}{n-i}\,\alpha^{**(1)}_{i:n}\right]. \qquad (6.28)$$

Proof. For $1 \le i \le n-2$, let us consider

$$\alpha^{**(1)}_{i:n} = E(Z^{*}_{i:n}\, Z^{*0}_{i+1:n})$$
$$= \frac{n!}{(i-1)!(n-i-1)!}\int_{Q_1}^{P_1} z_1 \left\{F^{**}(z_1)\right\}^{i-1} f^{**}(z_1)\, K(z_1)\, dz_1, \qquad (6.29)$$

where

$$K(z_1) = \int_{z_1}^{P_1} \left\{1-F^{**}(z_2)\right\}^{n-i-1} f^{**}(z_2)\, dz_2 . \qquad (6.30)$$

Upon using (6.6) in (6.30) and integrating by parts, we get

$$K(z_1) = (2P-1)\left[(n-i)\int_{z_1}^{P_1} z_2 \left\{1-F^{**}(z_2)\right\}^{n-i-1} f^{**}(z_2)\,dz_2 - z_1\left\{1-F^{**}(z_1)\right\}^{n-i}\right]$$

$$-(P-Q)\left[(n-i+1)\int_{z_1}^{P_1} z_2 \left\{1-F^{**}(z_2)\right\}^{n-i} f^{**}(z_2)\,dz_2 - z_1\left\{1-F^{**}(z_1)\right\}^{n-i+1}\right]$$

$$+ P_2\left[(n-i-1)\int_{z_1}^{P_1} z_2 \left\{1-F^{**}(z_2)\right\}^{n-i-2} f^{**}(z_2)\,dz_2 - z_1\left\{1-F^{**}(z_1)\right\}^{n-i-1}\right].$$

Upon substituting the above expression of $K(z_1)$ in (6.29) and simplifying the resulting equation, we derive the recurrence relation in (6.28).

<u>Relation 6.11</u>: For $n \geq 2$,

$$\alpha^{**}_{n-1,n:n+1} = \alpha^{**(2)}_{n-1:n+1} + \frac{(n+1)}{2(P-Q)}\left[nP_2\left\{P_1\alpha^{**(1)}_{n-1:n-1} - \alpha^{**(2)}_{n-1:n-1}\right\}\right.$$

$$\left. + (2P-1)\left\{\alpha^{**}_{n-1,n:n} - \alpha^{**(2)}_{n-1:n}\right\} - \alpha^{**(1)}_{n-1:n}\right]. \qquad (6.31)$$

<u>Proof</u>. For $n \geq 2$, let us consider

$$\alpha^{**(1)}_{n-1:n} = E(Z^{*}_{n-1:n}\, Z^{*0}_{n:n})$$

$$= n(n-1)\int_{Q_1}^{P_1} z_1 \left\{F^{**}(z_1)\right\}^{n-2} f^{**}(z_1)\, K(z_1)\, dz_1, \qquad (6.32)$$

where

$$K(z_1) = \int_{z_1}^{P_1} f^{**}(z_2)\,dz_2\,. \tag{6.33}$$

Upon using (6.6) in (6.33) and integrating by parts, we get

$$K(z_1) = (2P-1)\Big[\int_{z_1}^{P_1} z_2\, f^{**}(z_2)\,dz_2 - z_1\Big\{1-F^{**}(z_1)\Big\}\Big]$$

$$-(P-Q)\Big[2\int_{z_1}^{P_1} z_2\Big\{1-F^{**}(z_2)\Big\}f^{**}(z_2)\,dz_2 - z_1\Big\{1-F^{**}(z_1)\Big\}^2\Big]$$

$$+ P_2\,(P_1 - z_1).$$

Upon substituting the above expression of $K(z_1)$ in (6.32) and simplifying the resulting equation, we derive the recurrence relation in (6.31).

<u>Relation 6.12</u>: For $n \geq 2$,

$$\alpha^{**}_{2,3:n+1} = \alpha^{**(2)}_{3:n+1} + \frac{n+1}{2(P-Q)}\Big[\alpha^{**(1)}_{2:n} - nQ_2\Big\{\alpha^{**(2)}_{1:n-1} - Q_1\alpha^{**(1)}_{1:n-1}\Big\}$$

$$- (1-2Q)\Big\{\alpha^{**(2)}_{2:n} - \alpha^{**}_{1,2:n}\Big\}\Big]\,. \tag{6.34}$$

Proof. For $n \geq 2$, let us start with

$$\alpha_{2:2}^{**(1)} = E(Z_{1:2}^{*0}\, Z_{2:2}^{*})$$

$$= 2\int_{Q_1}^{P_1} z_2\, f^{**}(z_2)\, K(z_2)\, dz_2, \tag{6.35}$$

where

$$K(z_2) = \int_{Q_1}^{z_2} f^{**}(z_1)\, dz_1\,. \tag{6.36}$$

Upon using (6.5) in (6.36) and integrating by parts, we get

$$K(z_2) = Q_2(z_2 - Q_1) + (1-2Q)\left\{z_2\, F^{**}(z_2) - \int_{Q_1}^{z_2} z_1\, f^{**}(z_1)\, dz_1\right\}$$

$$-\,(P-Q)\left[z_2\left\{F^{**}(z_2)\right\}^2 - 2\int_{Q_1}^{z_2} z_1\, F^{**}(z_1)\, f^{**}(z_1)\, dz_1\right].$$

Upon substituting the above expression of $K(z_2)$ in (6.35) and simplifying the resulting equation, we derive the recurrence relation in (6.34).

Relation 6.13: For $2 \leq i \leq n-1$,

$$\alpha_{i+1,i+2:n+1}^{**} = \alpha_{i+2:n+1}^{**(2)} + \frac{(n+1)}{(i+1)(P-Q)}\left[\frac{1}{i}\,\alpha_{i+1:n}^{**(1)} - \frac{nQ_2}{i}\left\{\alpha_{i:n-1}^{**(2)} - \alpha_{i-1,i:n-1}^{**}\right\}\right.$$

$$\left. -\,(1-2Q)\left\{\alpha_{i+1:n}^{**(2)} - \alpha_{i,i+1:n}^{**}\right\}\right]. \tag{6.37}$$

Proof. For $2 \le i \le n-1$, let us consider

$$\alpha^{**(1)}_{i+1:n} = E(Z^{*0}_{i:n} Z^{*}_{i+1:n})$$

$$= \frac{n!}{(i-1)!(n-i-1)!} \int_{Q_1}^{P_1} z_2 \left\{1-F^{**}(z_2)\right\}^{n-i-1} f^{**}(z_2)\, K(z_2)\, dz_2, \tag{6.38}$$

where

$$K(z_2) = \int_{Q_1}^{z_2} \left\{F^{**}(z_1)\right\}^{i-1} f^{**}(z_1)\, dz_1 . \tag{6.39}$$

Upon using (6.5) in (6.39) and integrating by parts, we get

$$K(z_2) = Q_2\left[z_2 \left\{F^{**}(z_2)\right\}^{i-1} - (i-1) \int_{Q_1}^{z_2} z_1 \left\{F^{**}(z_1)\right\}^{i-2} f^{**}(z_1)\, dz_1\right]$$

$$+ (1-2Q)\left[z_2 \left\{F^{**}(z_2)\right\}^{i} - i\int_{Q_1}^{z_2} z_1 \left\{F^{**}(z_1)\right\}^{i-1} f^{**}(z_1)\, dz_1\right]$$

$$- (P-Q)\left[z_2 \left\{F^{**}(z_2)\right\}^{i+1} - (i+1)\int_{Q_1}^{z_2} z_1 \left\{F^{**}(z_1)\right\}^{i} f^{**}(z_1)\, dz_1\right] .$$

Upon substituting the above expression of $K(z_2)$ in (6.38) and simplifying the resulting equation, we derive the recurrence relation in (6.37).

In particular, by setting i=n–1 in (6.37), we obtain the recurrence relation

$$\alpha^{**}_{n,n+1:n+1} = \alpha^{**(2)}_{n+1:n+1} + \frac{n+1}{n(P-Q)}\left[\frac{1}{n-1}\,\alpha^{**(1)}_{n:n} - \frac{nQ_2}{n-1}\left\{\alpha^{**(2)}_{n-1:n-1} - \alpha^{**}_{n-2,n-1:n-1}\right\}\right.$$

$$\left. - (1-2Q)\left\{\alpha^{**(2)}_{n:n} - \alpha^{**}_{n-1,n:n}\right\}\right], \qquad n\geq 3. \qquad (6.40)$$

It should be mentioned here that by starting with the result that $\alpha^{**}_{1,2:2}=\alpha^{**2}_{1:1}$ (Govindarajulu, 1963; Joshi, 1971), one may employ Relations 6.8 – 6.13 to compute all the immediate upper–diagonal product moments, viz., $\alpha^{**}_{i,i+1:n}$ $(1\leq i\leq n-1)$, in a simple recursive way for all sample sizes. As mentioned earlier in Section 4, this is sufficient for the evaluation of all the product moments as the remaining product moments, viz., $\alpha^{**}_{i,j:n}$ for $1\leq i<j\leq n$ and $j-i\geq 2$, may be determined by using the recurrence relation in (4.13). However, for the sake of completeness we present here some more recurrence relations satisfied by the general product moments. These results may be established by following exactly the same steps as used in proving Relations 6.8 – 6.13.

<u>Relation 6.14</u>: For $1\leq i\leq n-2$,

$$\alpha^{**}_{i,n:n+1} = \alpha^{**}_{i,n-1:n+1} + \frac{n+1}{2(P-Q)}\left[nP_2\left\{P_1\alpha^{**(1)}_{i:n-1} - \alpha^{**}_{i,n-1:n-1}\right\}\right.$$

$$\left. + (2P-1)\left\{\alpha^{**}_{i,n:n} - \alpha^{**}_{i,n-1:n}\right\} - \alpha^{**(1)}_{i:n}\right]. \qquad (6.41)$$

<u>Relation 6.15</u>: For $1\leq i<j\leq n-1$ and $j-i\geq 2$,

$$\alpha^{**}_{i,j:n+1} = \alpha^{**}_{i,j-1:n+1} + \frac{n+1}{(n-j+2)(P-Q)}\left[\frac{nP_2}{n-j+1}\left\{\alpha^{**}_{i,j:n-1} - \alpha^{**}_{i,j-1:n-1}\right\}\right.$$

$$\left. + (2P-1)\left\{\alpha^{**}_{i,j:n} - \alpha^{**}_{i,j-1:n}\right\} - \frac{1}{n-j+1}\,\alpha^{**(1)}_{i:n}\right]. \qquad (6.42)$$

Relation 6.16: For $3 \le j \le n$,

$$\alpha^{**}_{2,j+1:n+1} = \alpha^{**}_{3,j+1:n+1} + \frac{n+1}{2(P-Q)}\Big[\alpha^{**(1)}_{j:n} - nQ_2\Big\{\alpha^{**}_{1,j-1:n-1} - Q_1\alpha^{**(1)}_{j-1:n-1}\Big\}$$

$$- (1-2Q)\Big\{\alpha^{**}_{2,j:n} - \alpha^{**}_{1,j:n}\Big\}\Big] . \qquad (6.43)$$

Relation 6.17: For $2 \le i < j \le n$ and $j-i \ge 2$,

$$\alpha^{**}_{i+1,j+1:n+1} = \alpha^{**}_{i+2,j+1:n+1} + \frac{n+1}{(i+1)(P-Q)}\Big[\frac{1}{i}\,\alpha^{**(1)}_{j:n} - \frac{nQ_2}{i}\Big\{\alpha^{**}_{i,j-1:n-1}$$

$$- \alpha^{**}_{i-1,j-1:n-1}\Big\} - (1-2Q)\Big\{\alpha^{**}_{i+1,j:n} - \alpha^{**}_{i,j:n}\Big\}\Big] .$$

$$(6.44)$$

7. DETAILS OF AVAILABLE TABLES

We list below the tables that are currently available on order statistics and their moments.

a. Table of $100\alpha\%$ points (for α=0.50, 0.75, 0.90, 0.95, 0.975, 0.990) for all order statistics for sample sizes up to 10 and for extreme and central order statistics for sample sizes from 11 to 25 has been given by Gupta and Shah (1965);

b. Table of probability integrals of the sample range W_n for n=2 and 3 evaluated at w=0.20(0.20)1.00(0.50)4.00 has been given by Gupta and Shah (1965);

c. Table of means and standard deviations of order statistics for sample sizes up to 10 has been given by Birnbaum and Dudman (1963). Table of covariances of order statistics for sample sizes up to 10 has been given by Shah (1966). These two tables have been extended by Gupta, Qureishi and Shah (1967) for sample sizes up to 25. Recently, Balakrishnan and Malik (1990) have prepared tables of means, variances and covariances for sample sizes 50 and less in which the values are reported to ten decimal places;

d. By using the results presented in Section 6, Balakrishnan and Joshi (1983 b) have given tables of means, variances and covariances for the symmetrically truncated logistic distribution (with $Q=1-P=0.01, 0.05(0.05)0.20$) for sample sizes up to 10;

e. Balakrishnan (1985) has handled the half logistic distribution (case when $Q=\frac{1}{2}$ and $P=1$) and has presented tables of means, variances and covariances for sample sizes up to 15. He has also given tables of $100\alpha\%$ points (for $\alpha=0.01$, 0.05, 0.10(0.10) 0.90, 0.95, 0.99) for extreme order statistics for sample sizes up to 15. In addition, he has presented a table of modes of all order statistics for sample sizes up to 15.

8. PLACKETT'S APPROXIMATION

David and Johnson (1954) and Clark and Williams (1958) have developed some series approximations for moments of order statistics from an arbitrary continuous distribution. These have been developed by applying the probability integral transformation and then using the known moments of order statistics from the uniform distribution. Plackett (1958), instead, has used the logit transformation which transforms

an order statistics $T_{i:n}$ from an arbitrary continuous distribution into the order statistic $Z_{i:n}$ from the logistic $L(0,\frac{\pi^2}{3})$ distribution to develop some series approximations for the moments of $T_{i:n}$ in terms of the moments of logistic order statistics $Z_{i:n}$.

We have already seen in Section 3 that the moments and the cumulants of the logistic order statistics $Z_{i:n}$ are all available in explicit form. Now, by realizing that the logit transformation

$$Z = \ln\left\{\frac{F_T(t)}{1-F_T(t)}\right\} \tag{8.1}$$

transforms the order statistic $T_{i:n}$ from an arbitrary continuous distribution with cdf $F_T(t)$ into the logistic order statistic $Z_{i:n}$ and, therefore, expanding $T_{i:n}$ in a Taylor series about the point $E(Z_{i:n}) = \kappa_{i:n}^{*(1)}$, we derive

$$\begin{aligned} T_{i:n} = t^{(o)} &+ t^{(1)}\left[Z_{i:n} - \kappa_{i:n}^{*(1)}\right] + \tfrac{1}{2}\,t^{(2)}\left[Z_{i:n} - \kappa_{i:n}^{*(1)}\right]^2 \\ &+ \tfrac{1}{6}\,t^{(3)}\left[Z_{i:n} - \kappa_{i:n}^{*(1)}\right]^3 + \tfrac{1}{24}\,t^{(4)}\left[Z_{i:n} - \kappa_{i:n}^{*(1)}\right]^4 + \dots, \end{aligned} \tag{8.2}$$

where, for $n-i+1>i$,

$$\kappa_{i:n}^{*(1)} = -\left\{\frac{1}{i} + \frac{1}{i+1} + \dots + \frac{1}{n-i}\right\}$$

as derived in Section 3, and $t^{(j)}$ is the value of the j^{th} derivative of t with respect to Z at Z at $Z = \kappa_{i:n}^{*(1)}$. Now, by taking expectation on both sides of (8.2) and upon using

the exact and explicit expressions for the cumulants of logistic order statistics derived in Section 3, we obtain the series approximation

$$E\left[T_{i:n}\right] \simeq t^{(0)} + \frac{1}{2}\, t^{(2)}\, \kappa_{i:n}^{*(2)} + \frac{1}{6}\, t^{(3)}\, \kappa_{i:n}^{*(3)} + \frac{1}{24}\, t^{(4)} \left\{ \kappa_{i:n}^{*(4)} + 3\left[\kappa_{i:n}^{*(2)}\right]^2 \right\}. \tag{8.3}$$

The derivatives appearing as coefficients in the approximation in (8.3) are easy to obtain as in the case of approximations due to David and Johnson (1954) and Clark and Williams (1958). For example, for the standard normal distribution with probability density function $\phi(t)$ and cumulative distribution function $\Phi(t)$, we have

$$t^{(0)} = \Phi^{-1} \left\{ \exp\left[\frac{\kappa_{i:n}^{*(1)}}{1 + \kappa_{i:n}^{*(1)}} \right] \right\},$$

$$t^{(1)} = \Phi\,(1-\Phi)/\phi,$$

$$t^{(2)} = t^{(1)} \left\{ tt^{(1)} - (2\Phi - 1) \right\},$$

$$t^{(3)} = (\, t^{(1)})^3 + 2tt^{(1)}\, t^{(2)} + t^{(2)}\, (1 - 2\Phi) - 2t^{(1)}\Phi\, (1-\Phi),$$

and

$$t^{(4)} = 5(t^{(1)})^2\, t^{(2)} + t^{(3)} \left\{ 2tt^{(1)} - (2\Phi - 1) \right\}$$

$$+\, 2t^{(2)} \left\{ tt^{(2)} - 2\Phi\, (1-\Phi) \right\} + 2t^{(1)}\, (2\Phi - 1)\, \Phi\, (1 - \Phi).$$

The above given derivatives are all bounded. As pointed out by Blom (1958), suppose we include the first j–1 terms in the series expansion for $E(T_{i:n})$ obtained from (8.2), then the absolute value of the remainder after j–1 terms is at most

$$\frac{1}{j!} \max |t^{(j)}| \; E|Z_{i:n} - \kappa^{*(1)}_{i:n}|^j. \tag{8.4}$$

Since $E|Z_{i:n} - \kappa^{*(1)}_{i:n}|^{2j}$ is known and also that

$$\left\{ E|Z_{i:n} - \kappa^{*(1)}_{i:n}|^{2j-1} \right\}^{\frac{1}{2j-1}} \le \left\{ E|Z_{i:n} - \kappa^{*(1)}_{i:n}|^{2j} \right\}^{\frac{1}{2j}},$$

we will be able to present bounds to $E(T_{i:n})$ for all values of j.

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18. SUBJECT TERMS *(Continue on reverse if necessary and identify by block number)*
Order statistics, Logistic distribution, Single and product moments, Recurrence relations, Percentage points, Modal properties, Series approximations, Truncated distributions

19. ABSTRACT *(Continue on reverse if necessary and identify by block number)*
Order statistics from the logistic distribution have been studied and their moments and distributional properties have been discussed. Recurrence relations satisfied by the single and the product moments of order statistics are presented. Series expansions are given for moments of order statistics from an arbitrary distribution based on the above results. Generalizations of all these results to the truncated logistic distribution are also given. An updated bibliography of all papers and tables relevant to this subject is also included.

20. DISTRIBUTION / AVAILABILITY OF ABSTRACT
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